VARIATIONAL APPROACH TO THE CALCULATION OF A HIGH-FREQUENCY INDUCTION DISCHARGE

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We discuss a result, obtained in [1], that contradicts experiments. It is shown that the reason for the contradiction involves the nonapplicability of the principle of minimum entropy production to thermodynamical systems situated in an alternating electromagnetic field. For stationary regimes of high-frequency discharges we formulate a variational problem that is equivalent to the original system of equations. We give a solution of this problem in the "channel model" approximation of a discharge with and without taking account of radiation losses.

A high-frequency induction discharge at high pressure is governed by Maxwell's system of differential equations and the heat conduction equations (see, for instance, [2]). In [2-4] this system of equations has been integrated directly under specific assumptions concerning the geometry of the discharge and the character of the heat transfer. Attempts at the calculation of stationary regimes of a high-frequency discharge by a procedure based on the minimum-power variational principle, in analogy with the "channel model" of the column of an arc discharge [6], has been undertaken in [1, 5]. In [5], however, mathematical details of the method applied are lacking. From the results of [1] the conclusion follows that the existence of a high-frequency discharge with a thin skin-layer is impossible, which contradicts the experimental facts. This circumstance was noted in [7]. In the discussion of this contradiction in [8] the question was not resolved, as will become evident in what follows. Therefore, it is of interest to clarify the reason for the contradiction in [1] and to make a judgment as to the possibility of a variational approach to the calculation of stationary regimes of a high-frequency induction discharge.

1. In [1] the minimum-power variational principle has been employed for the calculation of the "channel model" of a high-frequency discharge: the power P dissipated in a stationary discharge is a minimum

$$\delta P = 0 \tag{1.1}$$

The relationship of condition (1.1) to the thermodynamical principle of minimum entropy production, allegedly established in [9] for an arc discharge, has served as the basis for the application of (1.1) to a high-frequency discharge. The question whether such a relationship exists has been discussed in [10], where it was shown that no such general principle of minimum power for an arc discharge exists, though in certain cases the application of the variational condition (1.1) can be justified. In particular, this condition can be used for the construction of the channel model of the column of an arc discharge. If we now postulate the validity of the principle of minimum entropy production for a high-frequency discharge, then it would appear that through considerations similar to those in [10] we can arrive at a conclusion as to the validity of the calculation of the channel model of a high-frequency discharge based on condition (1.1). However, it is at the same time necessary to take account of the following circumstance. In the calculation of the channel model of a discharge an important initial relationship is the condition for the overall energy balance in the discharge

$$P = N \tag{1.2}$$

where N is the energy loss due to heat conduction.

If condition (1.1) holds, then from (1.2) it follows that

$$\delta N = 0 \tag{1.3}$$

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It can be shown [1] that for a unit length of the channel

$$N = 2\pi\theta_0 (\ln R//r_0)^{-1} \tag{1.4}$$

where θ (T) is a function having a one-to-one relationship with the thermal conductivity of the working gas

$$\theta(T) = \int_0^T \lambda(\tau)\, d\tau, \qquad \theta_0 = \theta(T_0) \tag{1.5}$$

where λ(T) is the thermal conductivity, T_0 and r_0 are the temperature and radius of the channel respectively, R is the radius of the discharge chamber,

$$P = \pi I^2 n^2 r_0 F(\rho_0)(\sigma_0\delta_0)^{-1} \tag{1.6}$$

I is the amplitude of the current in the inductor, n is the number of inductor windings per unit length, $\sigma_0 = \sigma(\theta_0)$ is the electrical conductivity, $\delta_0 = c\,(2\,\pi\sigma_0\omega)^{-1/2}$ is the thickness of the skin-layer, c is the velocity of light in vacuo, ω is the angular velocity of the electromagnetic field, and

$$F(\rho_0) = \sqrt{2}\,\frac{\mathrm{ber}(\rho_0)\,\mathrm{ber}'(\rho_0) + \mathrm{bei}(\rho_0)\,\mathrm{bei}'(\rho_0)}{\mathrm{ber}^2(\rho_0) + \mathrm{bei}^2(\rho_0)}$$

$$\rho_0 = \sqrt{2}\, r_0/\delta_0$$

As possible values of r_0 and θ_0 must satisfy (1.2), variations in (1.1) and (1.3) must be carried out along the curve $\theta_0(r_0)$, determined by relation (1.2). Then it follows from (1.4) that for condition (1.3) to be satisfied by a small deviation $\delta\theta_0 \gtreqless 0$ from the true value there must be a corresponding deviation $\delta r_0 \lesseqgtr 0$. However, this is not always achievable for Joule dissipation power, as determined by Eq. (1.6). In fact, for $\rho_0 \gg 1$ the right-hand side of (1.6) approaches

$$P = \pi I^2 n^2 r_0 (\sigma_0\delta_0)^{-1} \sim r_0\sigma_0^{-1/2} \tag{1.7}$$

It follows from this that for condition (1.1) to be satisfied when the electrical conductivity $\sigma(\theta)$ always increases with temperature a positive variation $\delta\theta$ must correspond to a positive variation of δr_0 and vice versa, which contradicts (1.2)-(1.4). It was just this circumstance that led to the contradiction in [1]. Since as shown in [10], condition (1.1) is a consequence of the extremality of entropy production, the extent to which we apply the principle of minimum entropy production in the description of a high-frequency discharge should be analyzed.

2. In nonequilibrium thermodynamics [11] the variational principle of minimum entropy production is well known. Entropy production is regarded as a functional and the problem of finding extrema of this functional is posed. In a steady state of any thermodynamical system, entropy production is an extremum because the Lagrange–Éuler equations for the corresponding variational problem are the equations that describe completely the steady regime of this system. In this sense the principle of minimum entropy production is simply equivalent to a system of equations describing a steady state of a thermodynamical system. If equations emerge, as the result of a variation of the entropy production, that do not describe a steady process completely (for example, certain steady-state equations are missing), then the principle of minimum entropy production is invalid for such thermodynamical systems.

In particular, it has been shown in [9] that under specific assumptions for a thermodynamical system that is able to conduct an electric current and is situated in a potential electric field (for example, a constant-current arc) the result of solving the variational problem for the extremum of entropy production is the emergence of the Élenbaas–Geller energy equation and the steady-state equation for the conservation of charge. These equations, together with Ohm's law, which is brought into the thermodynamics of irreversible processes as a phenomenological equation, completely describe both the electrodynamics and the heat exchange of the column of a steady-state constant-current arc. Therefore, under certain conditions a variational treatment of the steady-state regime of an arc discharge, based on the principle of minimum entropy production, seems possible.

It is easily shown that in the calculation of the extremum of entropy production for a thermodynamical system in an alternating electromagnetic field the same equations emerge as in the case of a constant-current arc, and Maxwell's system of equations for a continuous medium, governing the interaction of the alternating electromagnetic field with the thermodynamical system, does not appear. Therefore the principle of minimum entropy production is invalid for a high-frequency discharge.

Now the meaning of the contradiction, originating in [1], becomes clear. Roughly speaking, power dissipation is determined mainly by the electrodynamics of the discharge, which does not derive from the principle of minimum entropy production.

3. For a correct variational description of a steady high-frequency discharge, it is necessary to construct some functional such that the condition for its extremum yields a complete description of the steady-state regime of the discharge. As the construction of a functional whose extremality would be equivalent to the complete system of Maxwell's equations and the equations of heat conduction, it is necessary to separate the electrodynamic part of the problem from the energetical part. Then a variational approach to the problem is found to be possible.

Consider the functional

$$V[\theta] = \int_0^R \left[\left(\frac{d\theta}{dr}\right)^2 - |E^*(r)|^2 \int_0^{\theta(r)} \sigma(\theta)\, d\theta\right] r\, dr \tag{3.1}$$

and let E* (r) be the actual (nonvaried) distribution of the complex amplitude of the intensity of the electric field that becomes established in the steady-state regime of a high-frequency discharge. We shall seek the extremum of this functional

$$\delta V = 0 \tag{3.2}$$

Condition (3.2) is equivalent to the Lagrange–Éuler equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \frac{1}{2}\sigma(\theta)\,|E^*(r)|^2 = 0 \tag{3.3}$$

with the boundary conditions

$$\frac{d\theta}{dr}\,\delta\theta = 0 \tag{3.4}$$

at r = 0, r = R.

Equation (3.3) is the equation for energy balance in a steady cylindrical high-frequency discharge at high pressure under the condition that heat transfer occurs only through radial heat conduction. The boundary conditions (3.4) also correspond to the case under consideration under the condition that the walls of the discharge chamber are maintained at a constant temperature, which can be set equal to zero with no restriction of generality.

Thus, as the extremum of the functional (3.1) for a fixed distribution E*(r) yields the actual steady temperature distribution, the latter can be found immediately from (3.2) by direct methods of variational calculus.

We shall show how to apply the proposed variational principle for the description of the channel model of a high-frequency discharge.

In the channel model of a discharge, radial distributions of temperature and electrical conductivity of the forms

$$\theta(r) = \begin{cases} \theta_0, & 0 < r < r_0 \\ \theta_0 (\ln R/r)(\ln R/r_0)^{-1}, & r_0 < r < R \end{cases}$$
$$\sigma(r) = \begin{cases} \sigma_0, & 0 < r < r_0 \\ 0, & r_0 < r < R \end{cases} \tag{3.5}$$

are considered.

The distributions (3.5) correspond to a cylindrical, electrically conducting channel of radius r_0 at a constant temperature θ_0, surrounded by a tubular zone ($r_0 < r < R$) through which heat release from the channel is accomplished.

Inserting (3.5) into (3.1), we obtain

$$V(r_0, \theta_0) = \theta_0^2 (\ln R/r_0)^{-1} - \left(\int_0^{\theta_0} \sigma(\theta)\, d\theta\right)\left(\int_0^{r_0} |E^*(r)|^2\, r\, dr\right) \tag{3.6}$$

Thus, in the channel model approximation the correct functional V[θ] is represented in the form of a function of two variables, r_0 and θ_0. Condition (3.2) is now equivalent to the relations

$$\frac{\partial V}{\partial \theta_0}\bigg|_{\substack{\theta_0=\theta_0^* \\ r_0=r_0^*}} = 0, \quad \frac{\partial V}{\partial r_0}\bigg|_{\substack{r_0=r_0^* \\ \theta_0=\theta_0^*}} = 0 \tag{3.7}$$

where θ_0^* and r_0^* are the actual channel parameters of the discharge, which are subject to determination by means of (3.7). We remark that in the variational method indicated here it is not required that the varied parameters θ_0 and r_0 be related to each other through condition (1.2), which was the direct cause of the contradiction in [1].

It follows from (3.6), (3.7) that

$$2\theta_0^* (\ln R/r_0^*)^{-1} - \sigma_0^* \int_0^{r_0^*} |E^*(r)|^2 r dr = 0 \tag{3.8}$$

$$\theta_0^{*2} (r_0^* \ln^2 R/r_0^*)^{-1} - r_0^* |E^*(r_0^*)|^2 \int_0^{\theta_0^*} \sigma(\theta) d\theta = 0 \tag{3.9}$$

If

$$P = \int_0^{r_0^*} \sigma_0^* \frac{|E^*(r)|^2}{2} 2\pi r dr \tag{3.10}$$

is taken into account, then with the aid of (1.4) relation (3.8) can be given a visualizable physical interpretation – this is the condition for the integrated energy balance of the discharge, similar to (1.2).

With the help of (3.8) and (3.10) relation (3.9) can be reduced to the form

$$\int_0^{\theta_0^*} \sigma(\theta) d\theta = \frac{P^2}{4\pi^2 r_0^{*2} |E^*(r_0^*)|^2} \tag{3.11}$$

For the case of a thin-skin layer ($r_0^* \gg \delta_0^*$), for which conditions for the validity of the channel model are fulfilled most accurately (in the discharge zone volumetric energy release is absent), the square of the electric field strength at the boundary of the channel is given by the formula (see, for instance, [12])

$$|E^*(r_0^*)|^2 = 2I^2 n^2 (\sigma_0^* \delta_0^*)^{-2} \tag{3.12}$$

Then, inserting (1.7) and (3.12) into (3.11), we obtain

$$\int_0^{\theta_0^*} \sigma(\theta) d\theta = {}^1/_2 \left(\frac{In}{2}\right)^2 \tag{3.13}$$

Relation (3.13) determines the channel temperature θ_0^* in its dependence on the number of ampere-turns of the inductor as affiliated with the known function $\sigma(\theta)$ and, with an accuracy up to a factor equal to two, it coincides with the exact integral obtained in [3]. If the electrical conductivity depends on the temperature according to Boltzmann's law $\sigma \sim \exp(-A/2kT)$ and $A \gg 2kT$ (A is the ionization potential of the working gas and k is Boltzmann's constant), the integral in (3.13) can be evaluated approximately (see [13]). As a result the solution, obtained by the variational method, coincides with logarithmic accuracy with the results of [3, 4], which are based on a direct integration of the equations governing the steady regime of a discharge.

4. The method described above makes it possible to take account of the influence of the emission of radiation on the parameters of a high-frequency discharge. In the volumetric radiation approximation the equation of energy balance for the steady-state regime of a high-frequency discharge has the form [2]

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \frac{\sigma(\theta)|E^*(r)|^2}{2} - \Phi(\theta) = 0 \tag{4.1}$$

where $\Phi(\theta)$ is the volumetric density of the power loss due to radiation.

It is easily verified that Eq. (4.1) is the Lagrange–Éuler equation for the variational problem on the extremum of the functional

$$U[\theta] = \int_0^R \left[\left(\frac{d\theta}{dr}\right)^2 - |E^*(r)|^2 \int_0^{\theta(r)} \sigma(\theta) d\theta + 2\int_0^{\theta(r)} \Phi(\theta) d\theta\right] r dr \tag{4.2}$$

where, as formerly, $E^*(r)$ is not varied.

Because of the sharp temperature dependence of $\Phi(\theta)$, emission of radiation takes place mainly out of the channel. Therefore it is natural to assume that

$$\Phi(\theta)=\begin{cases}\Phi(\theta_0)\equiv\Phi_0, & 0<r<r_0\\ 0, & r_0<r<R.\end{cases} \tag{4.3}$$

Using the approximation (4.3) and applying the procedure described in Sec. 3, we obtain

$$U[r_0,\theta_0]=\theta_0^2(\ln R/r_0)^{-1}-\left(\int_0^{\theta_0}\sigma(\theta)\,d\theta\right)\left(\int_0^{r_0}|E^*(r)|^2 r\,dr\right)+r_0^2\int_0^{\theta_0}\Phi(\theta)\,d\theta \tag{4.4}$$

It now follows from relation (3.7) that

$$2\theta_0^*(\ln R/r_0^*)^{-1}-\sigma_0^*\int_0^{r_0^*}|E^*(r)|^2 r\,dr+r_0^{*2}\Phi_0^*=0 \tag{4.5}$$

$$\theta_0^{*2}(r_0^*\ln^2 R/r_0^*)^{-1}-|E^*(r_0^*)|^2 r_0^*\int_0^{\theta_0}\sigma(\theta)\,d\theta+2r_0^*\int_0^{\theta_0^*}\Phi(\theta)\,d\theta=0 \tag{4.6}$$

It is clear that when radiation losses are neglected, relations (4.5) and (4.6) pass over to relations (3.8) and (3.9), respectively.

As the power loss per unit length of the channel due to radiation is

$$N_{\text{rad}}=\pi r_0^{*2}\Phi_0^* \tag{4.7}$$

in the channel model approximation, relation (4.5), like (3.8), represents the condition for overall balance of the energy of the discharge

$$P=N+N_{\text{rad}} \tag{4.8}$$

The system of transcendental equations (4.5), (4.6) determines the temperature θ_0^* and the radius r_0^* of the discharge as functions of the number of ampere-turns of the inductor.

In [8] the conclusion is reached that a satisfactory treatment of the channel model of a high-frequency discharge is impossible without taking account of energy release outside the channel. As is evident from the results of the present paper, construction of a channel model of a high-frequency discharge, even with radiation losses taken into account, is possible on the basis of a correct variational approach, in contrast to the approach followed in [1].

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